STEPS TOWARDS A PRAGMATIC PROTOGEOMETRY

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Bene speremus, hominum enim vestigia video

Aristippus

I

Mathematics is not a subject removed from common toil, but part and parcel of the system of our knowledge and of the sciences as a whole. It would be a truly extraordinary situation in the cosmos if it were otherwise. If it were, there would be straightaway a bifurcation to be explained. An adequate account of knowledge without a bifurcation is difficult enough, and next to impossible with one. Further, there seem to be no well-grounded reasons why mathematics should be thought to constitute one kind of knowledge and the other sciences another. What precisely is the difference here anyhow? Is it fundamental? And why should there be this difference? Not that there are not distinctions, of course. There always are between or among the sciences, but these should not, it would seem, be regarded as fundamental differences of kind. In any case, it is of interest to try to view mathematics as part of an integrated and comprehensive system rather than as something special or *sui generis*.

By 'mathematics' is meant here primarily arithmetic and geometry, together of course with whatever can be constructed in terms of these. And by 'arithmetic' one means that of the positive integers, perhaps including 0 (in which case we speak of the natural numbers). By 'geometry' is meant primarily here what is called 'protogeometry', which includes only notions and principles of a very basic kind enunciated before getting on too far into the specialized geometries, Euclidean, Riemannian, and so on, of a higher number of dimensions.

The term 'protegeometry' is adapted from Paul Lorenzen,[1] the term, note, but not the content —who speaks of *geometry*, *chronometry*, and *hylometry* as constituting the three branches of what he calls 'protophysics'. According to him, these three "are a-priori theories which make empirical measurement of space, time, and materia [material bodies] "possible". They have to be established before physics in the modern sense of an empirical science, with its hypothetical fields of forces, can begin." Of course physics can "begin" any way it wishes and physicists are free to proceed howsoever they best see fit. The point of protophysics is that it provides a "rational reconstruction" or "logical map" of what the physicist presupposes. Fundamental here are the integers as used in counting and measurement (with the principles of arithmetic), and of course geometric elements (with at least rudimentary principles governing them). Lorenzen's own characterization of protophysics is highly unsatisfactory for a variety of reasons.[2] It seems best then to begin on a somewhat new footing provided by event logic, about which more will be said in a moment.

II

The *sui generis* approach to mathematics is via set theory. Many alternative set theories are currently on the market clamoring for buyers. It is not clear that any of them are fully satisfactory, even in the eyes of their proponents. There is little agreement as to which are preferable, on purely cognitive grounds, and the proponents of any one are eloquent in proclaiming its special virtues. Opponents are equally eloquent in pointing out the *ad hoc* and artificial character of crucial axioms needed to protect the system from the well known paradoxes. Even with this artificiality, convincing proofs of

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[1] In his *Normative Logic and Ethics* (Bibliographisches Institut, Mannheim — Zürich: 1969), p. 60.

[2] See the author's "Truth and Its Illicit Surrogates," *Neue Hefte für Philosophie*, forthcoming, and "On Lorenzen's *Normative Logie and Ethics*" in *Events and Other Logico-Philosophical Papers*. (The Catholic University of American Press, Washington: to appear).

consistency are not easy to achieve, and it may well be maintained that none has been, or perhaps ever will be, forthcoming.

Why then bother about set theory at all? one may ask. To provide a foundation for mathematics, one could answer. Whether mathematics is in any genuine sense founded upon set theory, however, may be doubted. Mathematics is something quite different, it may be maintained, more constructivistic, more intuitionistic, more numerical, more intimately linked with the other sciences and concerned with supplying suitable methods and procedures to them. If mathematics is so viewed, the set-theoretic approach to its foundations is not appropriate.

It has been maintained that set theory is needed only in mathematics, not outside it. According to Father Bochénski, it "seems that . . . [higher level] functors and quantifiers occur *only* in formal sciences, and are not needed in [the] empirical sciences."[3] Outside mathematics, the theory of *virtual* classes and relations seems always to suffice wherever classes or sets are needed at all.[4]

It is usually thought that sets are essential in semantics, being presupposed fundamentally in Tarski's celebrated definition of the truth concept.[5] Simpler methods are known, however, for providing for truth foregoing such powerful devices.[6] And similarly for syntax and pragmatics. Hence there is no need for set theory as a foundation for the theory of truth and other areas of semiotic.

If one is sensitive to matters of ontic commitment, one may well be bothered by the ontic status of sets and allied objects.[7] Any use of them commits one to a vast realm of "abstract" objects of dubious character. Such commitment is not altogether welcome on philosophical grounds and is to be circumvented if possible. In any case it raises an undesirable problem that can otherwise be avoided. As good a maxim as any in philosophy is that *problems should not be multiplied beyond necessity*.

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[3] See his *The Problem of Universals*, with A. Church and N. Goodman (University of Notre Dame Press, Notre Dame: 1956). p. 42.

[4] See the author's *Belief, Existence, and Meaning* (New York University Press, New York: 1969), Chapter VI.

[5] See his "The Concept of Truth in Formalized Languages," in *Logic, Semantics, Metamathematics* (Clarendon Press, Oxford: 1956), pp. 152-278.

[6] See especially the author's *Truth and Denotation* (University of Chicago Press, Chicago: 1958).

[7] See especially W. V. Quine, *Word and Object* (The Technology Press of the Massachusetts Institute of Technology and John Wiley and Sons, New York and London: 1960) pp. 119f. and 241ff., and *Belief, Existence, and Meaning*, Chapter II.

Some think that set-theoretic methods are indispensable in the study of the "deep structure" of natural language. The so-called model-theoretic semantics of "possible worlds," under current exploration and favored by many, presupposes set theory in most fundamental ways. It is far from clear, however, that such semantics succeeds where simpler methods fail. The study of deep structure from a logico-semantical point of view is still in its infancy, and all manner of different approaches to it should surely be explored. In the long run, simpler methods usually carry the day over complicated ones, however, in the sciences as elsewhere. There is thus little reason to think that the new linguistics, when it comes to terms with modern logic, will provide any exception to this.

Still another argument against set theory will be presented below, in terms of *extrapolation*. The natural numbers and the basic geometric entities emerge naturally by extrapolation from familiar entities and experiences, it will be maintained. Classes and sets cannot arise in such fashion, it seems. Being *sui generis*, one has to make a tremendous mental leap to grasp them. They do not seem to arise in the system of our knowledge along with the rest of what we know, but must instead occupy a very special place. This need for special handling is precisely what is denied in the account of arithmetic and geometry to be given. The 'Fragile— Handle with Care' label need be attached only where the contents are unable to stand on their own in the rough and tumble of transit in the actual world.

III

In several previous papers alternative approaches to the problem of formulating an event logic have been explored.[8] The concerns in those papers were mainly ontic and much depended thus upon the choice of values for the variables. The preferred formulation seems to be that in which *events and events only* are taken as values for variables. The problem is then to show how all that there is can in a natural way be construed as an event or as a "logical construct" in terms of such. Let us review briefly how this may be done, by way of preamble, and then go on to reflect upon the foundations of protogeometry.

[8] See especially *Belief, Existence, and Meaning, Chapter* IX: *Logic, Language, and Metaphysics* (New York University Press, New York: 1971), Chapters VII and VIII; and "Events" in *Events and Other Logico-Philosophical Papers.*

Roughly speaking, event logic may be divided into the following parts. First, the usual, classical theory of truth-functions, quantifiers, and identity (as between events) is presupposed —a simple, applied, functional calculus of first order with identity, in the terminology of Church[9]— together with the theory of *virtual classes and relations* already mentioned. To this the *calculus of individuals*[10] is added, either with or without a *null* individual.[11] The use of the latter is highly convenient for many technical purposes, and hence it will be admitted here without qualms. The formulation of the calculus of individuals will be in terms of a primitive part-whole relation as between events. Each non-null event contains as parts *point events* as their minimal constituents. Again, the admission of point events is perhaps not absolutely necessary but is of especial interest for present purposes.

Next we turn to the theory of *event-descriptive predicates*. It is most important to distinguish between events and event kinds, the latter being in effect virtual classes of the former. A particular throw of a pair of dice is an event, whereas all throws of those dice constitute a kind.[12] A particular performance of a Beethoven Sonata is an event, but all performances of it are a kind. Event kinds are accommodated by means of event-descriptive predicates, namely, predicates which when applied to an event say of that event that it is of such and such a kind.

In the previous formulations of event logic, a kind of spatiotemporal topology was assumed, in terms of which a temporal before-than relation was introduced. Such a relation is useful for accommodating the ordinary differences of tense, and of course for handling time both in ordinary language and in the foundations of the sciences. For the present, however, no such topology is assumed. Spatiotemporal order and location are to be presumed handled in a somewhat different way in terms of protogeometry.

Events are construed very broadly here, so as to include all man-

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[9] A. Church, *Introduction to Mathematical Logic*, I (Princeton University Press, Princeton: 1956).

[10] See especially H. S. Leonard and N. Goodman, "The Calculus of Individuals and Its Uses," *The Journal of Symbolic Logic* 5 (1940): 45-55, and J. H. Woodger, *The Axiomatic Method in Biology* (Cambridge University Press, Cambridge: 1937), Appendix E by A. Tarski.

[11] See "Of Time and the Null Individual," *The Journal of Philosophy* LXII (1965): 723-736.

[12] Cf. R. Carnap, *The Logical Foundations of Probability* (University of Chicago Press, Chicago: 1950), p. 35.

ner of acts, processes, states (mental and physical), including linguistic acts and events of all kinds. In terms of the latter a kind of *pragmatized syntax* and *semantics* may be introduced. An especially important subclass of events is therefore that of *sign events* or *inscriptions*. Both the syntax and semantics are thus of the inscriptional kind. In terms of such, a theory of *counting* may be developed and this by easy extrapolation leads into a theory of arithmetic and of constructive real numbers.

IV

The positive integers have been with us for some centuries now, holding their little heads as proudly as can be. God is supposed to have made them, according to Kronecker, and all the rest of mathematics is the work of man. The view here, however, is that even the integers are human artifacts.

In a previous paper a form such as

(1) 'e_1 Crrlt e_2, e_3, F'

was introduced primitively to express that person e_1 correlates a string of stroke-marks '||||. . .|' with those parts of the object e_3 that are in the virtual class F.[13] Thus where the e_2 is a single mark, either e_3 would itself be in F or would contain one and only one part that is in F. Where e_2 consists of just two strokes, e_3 would not be in F but would contain exactly two distinct proper parts that are. And so on. In this way a theory of counting is developed that serves as a basis for arithmetic. Expressions for the integers are defined in appropriate contexts, and the principles of arithmetic proved on suitable assumptions.

Arithmetic thus arises out of event logic in a natural way by extrapolation and without the use of any new type of object as values for variables. On this basis the theory of rational numbers, positive and negative, may be developed in familiar fashion, as well as a constructivistic theory of the real numbers. Whether anything more is really ever required in applications of mathematics in the sciences is a moot point. In any case, a very considerable portion of standard

[13] See "The Pragmatics of Counting," in *Events and Other Logico-Philosophical Papers*.

numerical mathematics may be accommodated on the kind of basis described.

V

Geometry like arithmetic is a product of human ingenuity. There are no geometrical entities "out there" waiting to be discovered. There are no eternal geometric truths independent of us awaiting formulation. On the contrary, it is we who construct the geometric entities in just such fashion as we please, doing so to serve some suitable human or noetic purpose. It is we who formulate geometric principles, in many alternative ways, adjusting them into total patterns of interest on their own account and often useful in practice. By a 'pattern' here one means what is usually called 'an axiom system'. The pattern incorporates implicitly the basic principles determinative of the particular kind of geometry.

If protogeometry like arithmetic is a branch of pragmatics, we must search for suitable basic pragmatical primitives in terms of which it may be expressed.

Geometry of the basic kinds postulates that there are such things as points, lines, planes, and so on, interrelated in various ways. For the moment let us consider only points. According to Euclid, in the very first definition of Book I of the *Elements*, "A point is that which has no parts, or which has no magnitude." By 'parts' here of course is meant 'proper parts'. Although Euclid fails to enunciate it clearly, there are assumed to be such things as points. Non-Euclidean systems make such an assumption also. If geometry is regarded as a part of a wider scheme, a good deal more needs to be said as to what points are and how they arrived at. And a good deal more needs to be said as to whether there are such entities. *As a matter of fact there are no points*, according to the theory to be put forward here, and that ends the matter. As convenient, idealized fictions, however, arrived at by extrapolation from what there actually is, they may be introduced and accommodated in suitable linguistic contexts. Just as there are no integers but merely strings of marks and suitable acts of correlation, so there are no points but merely certain pseudo-objects arrived at by suitable *Gedankenexperimente* or acts of extrapolation.

In event logic there are such things as *point events* and there is such a thing as a *null* event, as already remarked. Let 'e_1 P e_2' express that event e_1 is a *part* of event e_2. That e_1 is a *proper part* of e_2 may be expressed by 'e_1 PP e_2'. Clearly

$$(e_1)\ (e_2)\ (e_1 \text{ PP } e_2 \equiv (e_1 \text{ P } e_2\ .\ \sim e_2 \text{ P } e_1)).$$

Let 'Ne' express that the event e is null. Then

$$(\text{E}e)\ \text{N}e,$$

$$(e_1)\ (e_2)\ ((\text{N}e_1\ .\ \text{N}e_2) \supset\ e_1 = e_2),$$

$$(e)\ (\text{N}e \equiv \sim (\text{E}e_1)\ (\sim\text{N}e_1\ .\ e_1 \text{ P } e)),$$

$$(e)\ (\text{N}e \equiv \sim (\text{E}e_1) e_1 \text{ PP } e),$$

and

$$(e)\ (\text{N}e \equiv (e_1) e \text{ P } e_1).$$

There is such a thing as a null event, there is at most one null event, every null event is such that it has no part other than a null event, every null event has no proper part and conversely, and every null event is a part of every event and conversely.

Point events are non-null actual events having no proper parts other than the null event. Thus

'PtEv e' may abbreviate '$(\sim \text{N}e\ .\ (e_1)\ ((e_1 \text{ P } e\ .\ \sim \text{N}e_1) \supset e \text{ P } e_1))$'.

It seems reasonable in event logic to postulate that there are point events and to regard them as the most basic kind of non-null entity available. They are very like the little atomic triangles of Plato's *Timaeus* and the actual entities, the "merest puffs of experience," of Whitehead's *Process and Reality*.

VI

It might be thought that protogeometric points could be identified outright with point events, but this would not do. Point events have a little magnitude, to speak loosely, whereas points have none. Point events can be summed into larger objects, every non-null event being the *fusion* —in the sense of the calculus of individuals— of the virtual class of point events that are parts of it.

$$(e)\ (\sim \text{N}e \supset e = \text{Fu'}\, e' \ni (\text{PtEv } e'\ .\ e' \text{ P } e)).$$

(Here Fu'X is the *fusion* of the virtual class X).

Protogeometric points likewise can be summed, the result, however, having no more magnitude than the summands. Further, the cardinality of the PtEv's is less than that of the protogeometric points The latter is nondenumerable, the former at most denumerable, perhaps even finite.

Further kinds of pragmatic correlation relations may now be introduced in context to supplement (1) above, in particular now

(2) $\quad$ 'e_1 Crrlt_{Pt} e_2, e_3'.

This is to express that person e_1 correlates the event e_2 (regarded as a "point") with the PtEv e_3 as the "parent entity". Where 'Per e_1' expresses that e_1 is a person, clearly

$$(e_1)\ (e_2)\ (e_3)\ (e_1\ \text{Crrlt}_{\text{Pt}}\ e_2, e_3 \supset \ (\text{Per}\ e_1\ .\ \text{PtEv}\ e_3)).$$

Let 'e_1 Acpt e_2' express that e_1 accepts the sentence e_2 of L, and let 'e_1 Ref e_2, e_3' express that e_1 uses the sign event e_2 to refer to e_3.[14] Then,

(D1) 'e_1 Crrlt_{Pt} e_2, e_3' may abbreviate '(Per e_1 . PtEv e_3. (Ee') (Ee'') (Ee_4) (e_1 Ref e', e_2 . e_1 Ref e'', e_3 . e_1 Acpt e_4 . Γe_4)',

where 'Γe_4' expresses that e_4 is an inscription of the shape '(' followed by e' followed by 'PP' followed by e'' followed by '.' followed by '$\sim$(Ee_5)e_5PP' followed by e' followed by ')'. (This clause merely spells out the structural description of the shape of the sentence that person e_1 accepts.[15])

Although this definition looks somewhat laborious, actually it is quite simple. It defines (2) above as holding when person e_1 accepts a sentence to the effect that the entity or "point" e_2 is a proper part of point event e_3 but itself has no proper part. By accepting such a sentence e_1 is deliberately propagating a fiction. He is "imagining" a situation that might obtain. He takes or regards the entity e_2 here as an imagined point.

What is a point then? Let

(D2) $\quad$ '$\text{Pt}_{e'}$ e' abbreviate '(Ee_1)e' Crrlt_{Pt} e, e_1',

so that a point is merely an entity imagined or accepted by someone to be a proper part of some PtEv but having no proper part itself.

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[14] For further discusion of 'Ref', see the author's "On Truth, Reference, and Acts of Utterance," in *Events Etc.*

[15] On structural descriptions within inscriptional semantics, see *Truth and Denotation*, p. 247.

(D2) might strike one as providing a very strange conception of a point. It puts points, and therewith geometry, under a propositional attitude, so to speak. Further, all points deliberately become fictionalized. Still, there are such points if there are geometers to entertain them. In fact there are as many such points as any geometer cares to entertain. (In fact

$$(\mathrm{E}e')\ (\mathrm{Per}\ e'\ .\ \mathrm{Nc}'e\ni \mathrm{Pt}_{e'}\ e = \mathcal{N}_1),$$

if suitable means for expressing this fact are at hand.)[15b] And although the sentence e_4 of (D1) is false, it may be true that the geometer entertains it. Thus true sentences of geometry may be formulated even though the sentence under the propositional attitude is false.

The notion of a point has some kinship with Whitehead's notion of a coordinate division, a fundamental notion of his theory of extensive connection.[16]

VII

Suppose now that the foregoing, or something like it, gives a reasonable notion of a point. How do we go on to lines, planes, and so on?

Another kind of correlation may now be introduced, this time as a primitive. Let

(3) $\quad$ '$e\ \mathrm{Crrlt}_{\mathrm{Bet}}\ e_1, e_2, e_3$'

express primitively that person e correlates some point of PtEv e_1 with some point of PtEv e_2 to form a line segment between them with some point of PtEv e_3 lying between these two points.

Let now, to simplify notation,

'$\sim e_4 =_e e_5$' be short for '$(\mathrm{E}e_2)\ (\mathrm{E}e_3)\ (\mathrm{E}e')\ (e\ \mathrm{Ref}\ e_2, e_4\ .\ e\ \mathrm{Ref}\ e_3, e_5\ .\ \Gamma e'\ .\ e\ \mathrm{Acpt}\ e')$',

where '$\Gamma e'$' expresses that e' is an inscription consisting of a '$\sim$' followed by the sign event e_2 followed by an '$=$' followed by e_3. And similarly for other forms of sentence. Further, to remind us that it is always a geometer, or at least *homo qua geometres*, who is

[15b] Here N'X is the cardinal number of the virtual class X.

[16] See the author's "On Coordinate Divisions in Whitehead's Theory of Extensive Connection," *Process Studies,* to appear, and pp. 436-438 of *Process and Reality* itself.

the person involved, let 'g' be used throughout in place of 'e' to refer to him.

Immediately then it is to hold that

$(g)\ (e_1)\ (e_2)\ (e_3)\ (g\ \mathrm{Crrlt_{Bet}}\ e_1, e_2, e_3 \supset (\mathrm{Per}\ g\ .\ \mathrm{PtEv}\ e_1\ .\ \mathrm{PtEv}\ e_2\ .\ \mathrm{PtEv}\ e_3\ .\ (\mathrm{E}e_4)\ (\mathrm{E}e_5)\ (\mathrm{E}e_6)\ (g\ \mathrm{Crrlt_{Pt}}\ e_4, e_1\ .\ g\ \mathrm{Crrlt_{Pt}}\ e_5, e_2\ .\ g\ \mathrm{Crrlt_{Pt}}\ e_6, e_3\ .\ \sim e_4 =_g e_5\ .\ \sim e_5 =_g e_6\ .\ \sim e_4 =_g e_6))),$

and

$(g)\ (e_1)\ (e_2)\ (e_3)\ (g\ \mathrm{Crrlt_{Bet}}\ e_1, e_2, e_3 \supset g\ \mathrm{Crrlt_{Bet}}\ e_3, e_2, e_1).$

Let

'$e\ \mathrm{Seg}_g\ e_1, e_2$' abbreviate '$(\mathrm{Pt}_g\ e_1\ .\ \mathrm{Pt}_g\ e_2\ .\ \sim e_1 =_g e_2\ .\ (e')\ (\mathrm{Pt}_g\ e' \supset (e'\ \mathrm{PP}_g\ e \equiv (e' =_g e_1\ \mathrm{v}\ e' =_g e_2\ \mathrm{v}\ (\mathrm{E}e_3)\ (\mathrm{E}e_4)\ (\mathrm{E}e_5)\ (g\ \mathrm{Crrlt_{Bet}}\ e_3, e', e_4\ .\ g\ \mathrm{Crrlt_{Pt}}\ e_1, e_3\ .\ g\ \mathrm{Crrlt_{Pt}}\ e_2, e_4\ .\ g\ \mathrm{Crrlt_{Pt}}\ e', e_5)))))$'.

('$e'\ \mathrm{PP}_g\ e$' here expresses that g takes e' to be a proper part of e). The definiendum expresses that g takes e as a line *segment* with e_1 and e_2 as endpoints. Then also

$(g)\ (e_1)\ (e_2)\ (e_3)\ (g\ \mathrm{Crrlt_{Bet}}\ e_1, e_2, e_3 \supset (\mathrm{E}e)\ (\mathrm{E}e_4)\ (\mathrm{E}e_5)\ (\mathrm{E}e_6)\ (g\ \mathrm{Crrlt_{Pt}}\ e_4, e_1\ .\ g\ \mathrm{Crrlt_{Pt}}\ e_5, e_2\ .\ g\ \mathrm{Crrlt_{Pt}}\ e_6, e_3\ .\ e\ \mathrm{Seg}_g\ e_4, e_6\ .\ e_4\ \mathrm{PP}_g\ e\ .\ e_5\ \mathrm{PP}_g\ e\ .\ e_6\ \mathrm{PP}_g\ \mathrm{e})).$

These three principles express familiar laws: that if a point b is between a and c, then a, b, and c are distinct points on a line segment and b is also between c and a.

Note that it is not required that, where $g\ \mathrm{Crrlt_{Bet}}\ e_1, e_2, e_3$, the PtEv's e_1, e_2, and e_3 are distinct. They may or may not be. The correlated points, however, must be.

Note also that where $e\ \mathrm{Seg}_g\ e_1, e_2$, only the existence of e is required, not necessarily uniqueness.

What now is a line? Let

'$e\ \mathrm{L}_g\ e_1, e_2$' abbreviate '$(\mathrm{Pt}_g\ e_1\ .\ \mathrm{Pt}_g\ e_2\ .\ \sim e_1 =_g e_2\ .\ (e')\ ((\mathrm{E}e_3)\ (\mathrm{E}e_4) e'\ \mathrm{Seg}_g\ e_3, e_4 \supset (e'\ \mathrm{PP}_g\ e \equiv (\mathrm{E}e_3)\ (\mathrm{E}e_4)\ (\mathrm{E}e_5)\ (\mathrm{E}e_6)\ (e'\ \mathrm{Seg}_g\ e_1, e_3\ .\ g\ \mathrm{Crrlt_{Pt}}\ e_1, e_4\ .\ g\ \mathrm{Crrlt_{Pt}}\ e_2, e_5\ .\ g\ \mathrm{Crrlt_{Pt}}\ e_3, e_6\ .\ (g\ \mathrm{Crrlt_{Bet}}\ e_4, e_5, e_6\ \mathrm{v}\ g\ \mathrm{Crrlt}\ e_4, e_6, e_5))))))$'

A line determined by points e_1 and e_2 according to this definition, is taken to consist of all segments determined by points e_1 to e_3 so to speak, where the point e_2 is taken to be between the points e_1 and e_3 or the point e_1 is taken to be between e_3 *and* e_2.

The foregoing principle may now be strengthened to read:

$(g)\ (e_1)\ (e_2)\ (e_3)\ (g\ \mathrm{Crrlt}_{\mathrm{Bet}}\ e_1, e_2, e_3 \supset (\mathrm{E}e)\ (\mathrm{E}e_4)\ (\mathrm{E}e_5)\ (\mathrm{E}e_6)\ (g\ \mathrm{Crrlt}_{\mathrm{Pt}}\ e_4, e_1\ .\ g\ \mathrm{Crrlt}_{\mathrm{Pt}}\ e_5, e_2\ .\ g\ \mathrm{Crrlt}_{\mathrm{Pt}}\ e_6, e_3\ .\ e\ \mathrm{L}_g\ e_4, e_6\ .\ e_4\ \mathrm{PP}_g\ e\ .\ e_5\ \mathrm{PP}_g\ e\ .\ e_6\ \mathrm{PP}_g\ e))$.

This states in effect that if a point *b* is between *a* and *c*, then *a*, *b*, and *c* all fall on a line.

Two lines intersect each other if they have a point in common. Thus

'$e_1\ \mathrm{Intsct}_g\ e_2$' abbreviates '$(\mathrm{E}e_3)\ (\mathrm{E}e_4)\ (\mathrm{E}e_5)\ (\mathrm{E}e_6)\ (\mathrm{E}e)\ (\mathrm{Pt}_g\ e\ .\ e_1\ \mathrm{L}_g\ e_3, e_4\ .\ e_2\ \mathrm{L}_g\ e_5, e_6\ .\ e\ \mathrm{P}_g\ e_1\ .\ e\ \mathrm{P}_g\ e_2)$'.

What now is a plane as determined by three non-colinear points? It may be taken as consisting of the three lines determined by those three points together with all lines that intersect them. Thus

'$e\ \mathrm{Pl}_g\ e_1, e_2, e_3$' may abbreviate '$(\mathrm{Pt}_g\ e_1\ .\ \mathrm{Pt}_g\ e_2\ .\ \mathrm{Pt}_g\ e_3\ .\ \sim e_1 =_g e_2\ .\ \sim e_2 =_g e_3\ .\ \sim e_1 =_g e_3\ .\ \sim (\mathrm{E}e_4)\ (\mathrm{E}e_5)\ (\mathrm{E}e_6)\ (e_4\ \mathrm{L}_g\ e_5, e_6\ .\ e_1\ \mathrm{PP}_g\ e_4\ .\ e_2\ \mathrm{PP}_g\ e_4\ .\ e_3\ \mathrm{PP}_g\ e_4)\ .\ (\mathrm{E}e_4)\ (\mathrm{E}e_5)\ (\mathrm{E}e_6)\ (e_4\ \mathrm{L}_g\ e_1, e_2\ .\ e_5\ \mathrm{L}_g\ e_2, e_3\ .\ e_6\ \mathrm{L}_g\ e_1, e_3\ .\ (e')\ ((\mathrm{E}e_7)\ (\mathrm{E}e_8)\ e'\ \mathrm{L}_g\ e_7, e_8 \supset (e'\ \mathrm{PP}_g\ e \equiv (e' =_g e_4 \vee e' =_g e_5 \vee e' =_g e_6 \vee (\mathrm{E}e_7)\ (\mathrm{E}e_8)\ (e'\ \mathrm{L}_g\ e_7, e_8\ .\ (e'\ \mathrm{Intsct}_g\ e_4 \vee e'\ \mathrm{Intsct}_g\ e_5 \vee e'\ \mathrm{Intsct}_g\ e_6)))))))$'.

Clearly suitable principles may now be laid down interrelating those various notions as needed for special purposes.

VIII

Whatever the defects of the foregoing may be, enough has been said to enable us to see how protogeometry emerges from event logic, on the one hand, and leads on to the development of geometry proper, on the other. Some of the principles given might be suitable to take as axioms. They are akin in fact to some of Hilbert's axioms for Euclidean geometry.[17] However, nothing like a complete axiomatization is attempted here. In fact, an axiomatization would belong to geometry itself rather than to protogeometry. Protogeometry is concerned merely with the forms of expression allowed, the vocabulary, and with how this vocabulary may be accomodated in the

[17] D. Hilbert, *The Foundations of Geometry*, 3rd ed. (Open Court Publishing Co., LaSalle, Ill.: 1938).

underlying event logic. Protogeometry is thus of the utmost importance philosophically. It should help us to understand what geometry is and how it relates to other areas of our knowledge. It shows us that geometry is not a subject apart, not "pure," but always applied to the actual world we inhabit. It shows us that geometry develops always under a propositional attitude and seeks to make explicit what this attitude is. A geometry is a *Gedankenexperiment*, nothing less, nothing more.

Hilbert's axioms, it will be recalled, divide into five parts: axioms of connection, axioms of order, the axiom of parallels, axioms of congruence, and the axiom of continuity (or the Archimedean axiom). Axioms of the first three kinds may readily be stated within protogeometry in the vocabulary given. Those would determine affine geometry, a Euclidean geometry without congruence and without a metric. (There are of course alternatives to the axiom of parallels that may be taken if the purpose at hand warrants — an understatement indeed.) The problem of how best to introduce a metric here is of course the problem of how to introduce methods of measurement in terms of correlational acts. Measurement is in fact merely one more species of correlation. Numbers have been introduced above only in contexts of counting. Measurement is a mere refined species of counting, of lengths, of mass, of force, and so on. In any case, congruence should be readily definable once suitable rules for measurement have been given. The remainder of Hilbert's axioms could then be formulated.

Note that the only specifically geometric primitive required in the above is '$\mathrm{Crrlt}_{\mathrm{Bet}}$'. It might seem a defect that such a primitive is required. Somehow it ought to be definable, it would seem, in terms already available. The relation $\mathrm{Crrlt}_{\mathrm{Bet}}$ involves a notion of betweenness and hence of order, however, and it is not clear how such a notion can be achieved otherwise than by means of adopting a primitive. If a temporal topology were assumed, it might be possible to define '$\mathrm{Crrlt}_{\mathrm{Bet}}$' in its terms. This however, is a matter beyond the confines of the present discussion.

The methods of protogeometry proceed by extrapolation, already mentioned above. In other words, the values for variables are just those of the underlying event logic with no additions. The extrapolations occur only under a propositional attitude. The interesting circumstance for the philosophy of geometry is that everything we wish to say is said under the attitude. There are no real points, lines, planes, and so on, but they may be talked about as imagined or fictitious or "theoretical" entities or constructs. Further, as the definition of 'Pt' shows, those constructs can be given in most natural ways just as though they were values for variables.

The question arises as to whether such extrapolation could also be made to work for the theoretical constructs of physics and other sciences. If so, protogeometry would become a branch of protophysics. There is also kinship with instrumentalism. The instrumentalist view of scientific methodology has emphasized all along that theoretical principles and entities are not to be taken at face value anyhow, but are more devices for giving explanations and predictions in terms of observed data.[18] In putting the theoretical principles and entities under a propositional attitude, a view akin to the instrumentalist one can perhaps be made more precise and explicit.

The only propositional attitude considered is that of acceptance, taken in the sense of provisional acceptation subject to improvement and correction. Is not this the attitude of the scientist anyhow to theoretical entities and principles? No finality of acceptance is required, no belief even, no knowledge — just the attitude of: let us assume it and see how it works out and improve and correct it as needed as we go along.

It was mentioned above that the method of extrapolation would not extend to set theory. The reason is that sets and allied entities are of an altogether new kind. Numbers, points, and so on, can be characterized in terms of the entities already available as values for variables, even if only under the attitude of acceptance. Sets, relations, and other such objects, it seems, must be handled rather as *values for a new kind of variable*. No other methods for formulating set theory have been developed, and may not even be possible. Further, a protogeometric point is in a real sense "interior" to its parent point event. Whatever a set is, however —and it is to be feared that no one has ever told us— it is "external" to its members. Perhaps all theoretical entities should be regarded as interior to the concrete entities from which they arise by extrapolation, at least microcosmic ones. In any case, it is by no means clear that a set theory could ever be formulated extrapolationally. Hence we have here one more telling argument against such theories.

It was mentioned above that protogeometry is concerned with fundamental notions and principles enunciated before getting too far into the special kinds of geometry. On the other hand, the fore-

[18] Cf. E. Nagel, *The Structure of Science* (Harcourt, Brace ard World, New York: 1961), Chapters 4, 5, and 6.

going has been modelled primarily on Euclidean geometry of three dimensions, as in Hilbert's axiomatization. Strictly there are two stages in the development of protogeometry to be distinguished, those required say, for geometries of Euclidean, Riemannian, Lobatschewskian, or other types, on the one hand, and then notions and principles common to all of those, on the other. The latter can be arrived at only after some delineation of the former.

Strictly protogeometry should consist only of notions neutral as among the specialized geometries and thus of principles common to them. The axioms of the special geometries then can be characterized as certains patterns, in effect, patterns of acceptance, made for certain purposes. A specialized geometry is thus something like a suit of clothes, to be put on or taken off as best befits the ocassion. The garments, however, must all be make out of the same basic materials, modelled in high fashion or low as the style demands.

Let '$\Gamma_G\ g, e$' express that e is an inscription consisting of the conjunction of the axioms taken by g as constituting some geometry G. Scientific laws in sciences requiring this type of geometry become then of the form

$$\text{`}(e)\ (\Gamma_G\ g, e \supset \text{———})\text{'},$$

where '———' presupposes explicitly or implicitly principles of this type of geometry. That scientific laws can be expressed in such forms would be a fundamental tenet of instrumentalism as conceived here. The axioms of G are taken merely as hypotheses where needed, but are not regarded as true. They are merely accepted by the scientist g to perform the tasks he requires of them.

A kind of instrumentalism and constructivistic protogeometry can thus be made to go hand in hand. Hopefully the foregoing provide some useful steps, however tentative and inadequate, as a basis for further exploration and development in order to deepen our understanding of both. *Quod erat faciendum.*

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