Diálogos, **59** (1992) pp. 121–134.

INTERDERIVABILITY OF SEEMINGLY UNRELATED MATHEMATICAL STATEMENTS AND THE PHILOSOPHY OF MATHEMATICS

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§ 1 The Problem

Interderivability of mathematical statements on the basis of certain assumptions or, briefly, mathematical equivalence of two statements in a determined area of mathematics is a relatively common phenomenon in the mathematical sciences. That two mathematical statement S and S' in general topology—or in group theory, or in number theory—are interderivable, is a familiar mathematical phenomenon. That a group G has property P if and only if it has property P', or that a topological space T has property Q if and only if it has property Q' seems completely natural both to the mathematician and to the philosopher of mathematics.

However, that mathematical statements of the most diverse areas of mathematics and apparently speaking about very different things, are interderivable, does not seem so natural. That a mathematical statement that speaks about well-ordered sets is interderivable, under Zermelo-Fraenkel set theory (from now on ZF),[1] with, e.g. a statement about vector spaces, or with a statement about topological spaces, or with a statement about the cardinality spectrum of models of sets of first order sentences, seems bizarre and at first sight highly implausible.

But precisely that seemingly implausible situation occurs in classical mathematics, and its philosophical significance has been neglected.

[1] From now on we will most frequently simply write 'interderivable' and 'mathematically equivalent' instead of 'interderivable under ZF' or 'mathematically equivalent under ZF'. Throughout this paper all four expressions are taken as synonyms.

Under ZF, the Ultrafilter Theorem, namely, the statement that says that every filter on a set can be extended to an ultrafilter, is interderivable, e.g. with the following three statements:[2]

(1) On each infinite set there is a two valued additive measure such that each singleton has measure zero.
(2) The product of any family of compact Hausdorff spaces is a compact Hausdorff space.
(3) *Compactness Theorem for first order logic*: A set S of first order sentences has a model if and only if every finite subset of S has a model.

For anyone acquainted with the concepts occurring in those four mathematically equivalent statements, it is clear that these statements speak about different and seemingly unrelated things.

A still more dramatic, but essentially similar, situation is that of the interderivability, under ZF, of the Axiom of Choice with many other mathematical statements in the most diverse areas of mathematics (and logic). The Axiom of Choice states that, given any family $\mathcal{F}$ of non-empty pairwise disjoint sets, there exists a function f such that for each set S in $\mathcal{F}$, $f(S) \in S$, i.e. the function selects from each set S in $\mathcal{F}$ a representative of S. As a way of further illustrating the interderivability phenomena between seemingly unrelated mathematical statements, let us consider the following list of a few of the many mathematical equivalents of the Axiom of Choice:

(1) *Well Ordering Theorem*: Every set can be well-ordered.
(2) *Zorn's Lemma*: If $\mathcal{F}$ is a family of sets such that the union of every chain $\mathcal{C} \subseteq \mathcal{F}$ is in $\mathcal{F}$, then $\mathcal{F}$ contains a maximal set under inclusion.
(3) Every lattice with a unit and at least another element has a maximal ideal.
(4) If $\mathcal{V}$ is a real vector space, then for every subspace $\mathcal{S}$ of $\mathcal{V}$ there is a subspace $\mathcal{S}'$ of $\mathcal{V}$ such that $\mathcal{S} \cap \mathcal{S}' = \{0\}$ and $\mathcal{S} \cup \mathcal{S}'$ generates $\mathcal{V}$.

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[2] Both for the equivalents of the Ultrafilter Theorem and for those of the Axiom of Choice, see G. H. Moore, *Zermelo's Axiom of Choice*, New York: Springer, 1982. See especially p. 328 for equivalents of the Ultrafilter Theorem and pp. 330–333 for equivalents of the Axiom of Choice. See also H. Rubin and J. E. Rubin's *Equivalents of the Axiom of Choice*, Amsterdam: North Holland, 1963 or *Equivalents of the Axiom of Choice*, II, Amsterdam: North Holland, 1985.

(5) *Löwenheim-Skolem-Tarski Theorem*: If a countable set of first order sentences has an infinite model, then it has a model of each infinite cardinality.

(6) *Tychonoff's Compactness Theorem*: The product of any family of compact topological spaces is a compact topological space.

For any one familiar with the concepts occurring in the Axiom of Choice and its six mathematical equivalents cited immediately above, it is clear that they all speak about very different things, e.g. about choice functions, well orderings, lattices, real vector spaces, compact topological spaces, and the cardinality spectrum of models of countable sets of first order sentences.

The problem that we would like to consider in this paper and that seems not to have received its due attention by philosophers of mathematics is whether such interderivability phenomena of seemingly unrelated mathematical statements have any relevance for the philosophy of mathematics or are completely neutral with respect to our choice of a particular philosophy of mathematics. More explictly, we ask if all philosophies of mathematics can fare well with respect to these phenomena.

§ 2 A Brief Examination of some Philosophies of Mathematics

It would be a formidable and rather boresome task to examine in detail the wide variety of philosophies of mathematics with respect to the problem in which we are interested here, namely, rendering philosophically intelligible the interderivability phenomena of seemingly unrelated mathematical statements. However, some brief remarks seem appropriate in order to motivate defenders of at least some philosophies of mathematics to try to assess such interderivability phenomena from the standpoint of their respective conceptions of the nature of mathematics. The burden of the proof is on their side. Of course, they can ignore that burden simply by rejecting those phenomena as not belonging to mathematics as they conceive it. But in this way they would be depriving mathematics of some of its philosophically most puzzling results.

(*a*) For some philosophies of mathematics, however, there is no other possibility than to explicitly reject or simply ignore such interderivability phenomena. A formalist philosophy of mathematics like the one timidly advocated by Paul Benacerraf at the end of his "What numbers could not be'",[3] which does not accept the existence of numbers, but only of number words—and, hence is not very different from the formalism criticized by Frege a hundred years ago—, would surely not accept the existence of, e.g. topological spaces, vector spaces, lattices or ultrafilters, but at best that of topological space words, vector space words, lattice words, etc., whatever that may mean. Even the formulation of the statements in the two lists of mathematically equivalent statements of §1 would seem extremely bizarre, and their mathematical equivalence would be completely unintelligible.

(*b*) Contemporary nominalists are probably ontologically more liberal than the (sorts of) formalists that we have been considering. According to Eberle,[4] contemporary nominalists postulate the existence of concrete individuals, but reject the existence of numbers, classes or other abstract entities. Of course, ultrafilters, lattices, vector spaces, topological spaces and such, as they are understood in classical mathematics, are clearly abstract entities. Nominalism has to deprive classical mathematics of some of its dearest parts, and at best try to construct a parallel mathematics—as Leśniewski did with his mereology as a possible substitute for set theory. But then the interderivability phenomena that we have been considering remain completely unintelligible for nominalism, since the statements that are proved to be mathematically equivalent speak about entities that are based on the classical notion of set and not on its mereological surrogate or on any other sort of nominalist substitute, which cannot have the same properties as sets. (E.g. in set theory—and in general topology, which is based on it—one distinguishes between a unit set and its unique member, whereas in mereology a mereological unit class is identitied with its unique member—or part, since mereology is a part-whole theory).

(*c*) Constructivism in mathematics is a particularly popular philosophical trend. The most basic difficulty with constructivism as a philosophy of mathematics, however, seems to lie in the term 'constructivism' itself. There does not seem to be any generally accepted precise meaning

[3] "What numbers could not be", in P. Benacerraf and H. Putnam (eds.), *Philosophy of Mathematics*, second edition, Cambridge: Cambridge University Press, 1983, pp. 272–294.

[4] Rolf A. Eberle, *Nominalistic Systems*, Dordrecht: Reidel, 1970, p. 6.

of that term. According, to Beeson,[5] by a constructivist philosophy of mathematics one understands a philosophy of mathematics based on the following two basic principles:

(1) to say that "x exists" means that there is a way to explicitly find x;
(2) the word "truth" does not have any a priori meaning, and a sentence is called 'true' just in case that a proof of it can be found.

The preceding characterization, however, is not sufficiently informative, since not only the philosophical preconceptions that underlie different constructivist schools can vary, but their notions of a mathematical proof can be very different and, thus, the results that the different constructivist schools consider mathematically sound can diverge essentially. Even in the most important constructivist school of this century, namely, intuitionism, there has been some disagreement concerning what should be considered mathematically sound.[6] Of course, if the philosophical preconceptions and the notion of a mathematical proof are too restricted, some of the entities spoken about in the statements belonging to either of the two lists of mathematically equivalent statements of §1 could not be constructed and many of the theorems related to those entities could not be obtained. E.g. the Axiom of Choice itself would certainly be considered mathematically inadmissible and the talk about different infinite cardinalities in the Löwenheim-Skolem-Tarski Theorem would be regarded as meaningless by many constructivists. Thus, if constructivism is understood in a too restrictive way, there is no hope of philosophically assessing the interderivability results under discussion. For such constructivisms those interderivability phenomena would be almost as unintelligible as for formalists and nominalists.

[5] Michael J. Beeson, *Foundations of Constructive Mathematics*, Berlin: Springer, 1985, p. 47. Beeson's constructivism follows that of Errett Bishop's *Foundations of Constructive Analysis*, New York: McGraw-Hill, 1967. See Ch. III of Beeson's book for other sorts of constructivism, to which one can add, e.g. I. Kant's remarks on mathematical method in *Kritik der reinen Vernunft*, 1781, second edition 1787, especially Part II: Transzendentale Methodenlehre, and M. Dummett's recent 'linguistic intuitionism' in *Elements of Intuitionism*, Oxford: Oxford University Press, 1977, and in 'The Philosophical Basis of Intuitionistic Logic', in P. Benacerraf and H. Putnam (eds.), op. cit. in footnote 3, pp. 97–129.

[6] See, e.g. E. W. Beth, *The Foundations of Mathematics*, Amsterdam: North Holland, 1965, Ch. 15, § 142, especially pp. 437–439.

We are not interested, however, in restricting in any way the notion of constructivism to win our case. Thus, let us suppose that there is a constructivism so liberal that it allows its defenders to acknowledge the existence of the same entities and to obtain the same theorems as classical mathematicians—with the only difference that some divergent methods are required, since they have to 'construct' the mathematical entities. Even in such a case, the interderivability results under discussion would remain a complete mystery for them. Since the entities spoken about in the two lists of mathematically equivalent statements of §1, namely, lattices, ultrafilters, topological spaces, vector spaces, cardinality spectra of models of sets of first order sentences, etc. differ considerably and have so diverging properties, even if all of them can be constructed and all corresponding theorems about them proved, the interderivability results would be philosophically as unintelligible for such a liberal constructivism as if, e.g. the statements 'Paris is the capital of France' were interderivable with the statement 'Plato was Aristotle's teacher.' What would be constructed by such a liberal constructivism are lattices, ultrafilters, topological spaces, etc., which are clearly very different mathematical entities. The interderivability of, e.g. the Löwenheim-Skolem-Tarski Theorem, Tychonoff's Theorem and Zorn's Lemma is in need of a philosophical assessment. But an adequeate assessment cannot be made if one assumes that all entities spoken about in such mathematical statements are constructed by (the community of) mathematical subjects. Hence, we have to conclude that no matter how liberal a constructivist philosophy of mathematics might be, it is incapable of an adequate philosophical assessment of the interderivability phenomena under discussion.

(*d*) After Frege's[7] and Husserl's[8] critiques of Mill's empiricism in mathematics and logic, one might have thought that empiricism concerning these disciplines would not reenter the philosophical scenario. Even logical empiricists clearly restricted their empiricist claims to other areas of science. In the last few decades, however, and probably under

[7] See G. Frege, ***Die Grundlagen der Arithmetik***, 1884, Centenarausgabe, edited and with an introduction by Christian Thiel, Hamburg: Meiner, 1986, especially §§ 7–10.

[8] See E. Husserl, *Logische Untersuchungen*, 1900–1901. Bd. I, in Husserliana, Vol XVIII, Den Haag: Nijhoff, 1975, especially Ch. V.

the influence of some remarks by Quine,[9] empiricism in logic and mathematics has rather shyly reappeared. Thus, it seems appropriate to examine how well does empiricism fare with respect to the problem that we have been discussing, namely, the philosophical assessment of the interderivability of seemingly unrelated mathematical statements.

Contrary to the other currents in the philosophy of mathematics briefly considered above, mathematical empiricism usually does not renounce from the outset to large parts of classical mathematics. The basic problem with empiricism of whatever sort is to formulate a convincing theory of how is it that we come to consider so highly abstract mathematical entities as ultraproducts, toposes and many others on a more or less thin empirical basis, and how is it that on such a basis we obtain mathematical knowledge about such entities. The burden of the proof is clearly on the side of mathematical empiricsts. Moreover, if the empirical data accepted are so thin as those of (the sort of) behaviourism presupposed by Quine,[10] there is no hope of completing the task. If the empirical basis is such as to allow the so-called indeterminacy of translation[11] and the so-called subdeterminacy of physical theories,[12] there is certainly no possibility of explaining our acquaintance with ultraproducts, topological spaces, algebras, etc. and our mathematical knowledge concerning them. Such entities and their properties are much farther away from any empirical basis than languages or physical theories. Moreover, our mathematical knowledge presupposes language (or some sort of symbolic system of representing concepts), and the collective mathematical knowledge of the mathematical community presupposes the translatability of mathematical texts. Our mathematical knowledge not only cannot be obtained from such a thin empirical basis, but—if Quine's indeterminacy thesis of translation is correct—is hardly compatible with it.

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[9] See, e.g. W. O. Quine, 'Two Dogmas of Empiricism', in *From a Logical Point of View*, 1953, second edition, Cambridge MA: Harvard University Press, 1961, pp. 20–46, especially pp. 42–46. See also his *Philosophy of Logic*, Englewood Cliffs: Prentice Hall, 1970, especially Ch. 7.

[10] See, e.g. W. O. Quine, *Word and Object*, Cambridge, MA: MIT Press, 1960.

[11] Ibid., especially Ch. II.

[12] See, e.g. 'Two Dogmas of Empiricism', p. 43.

Some mathematical empiricists, like Kitcher,[13] would probably say that the empirical basis does not have to be so thin and that one should allow the mathematical subject more freedom for constructing mathematical entities. Kitcher even speaks about an idealized mathematical subject.[14] In such a case, it seems pertinent to ask if this idealized mathematical subject operates on a purely empirical basis, i.e. without any sort of categorial device in Husserl's sense. Moreover, Kitcher says that mathematics is concerned with structures present in physical reality.[15] One should ask Kitcher to point to a physical structure that has some resemblance with, e.g. an ultraproduct, and one should urge him to construct such a mathematical entity on a purely empirical basis. One could continue arguing in this direction against Kitcher, who, e.g. says that arithmetic owes its truth to the structure of the world.[16] For even if the physical world were completely different from ours, mathematical and, hence, also arithmetical theorems would continue to be true.

However, we are not interested here in such more or less traditional arguments against empiricism. Let us assume, contrary to all available evidence, that mathematical empiricists succeed in constructing all entities of classical mathematics on a purely empirical basis—i.e. without any unacknowledged non-empirical tools—, and that they are capable of proving all theorems of classical mathematics. Even in such very improbable case, the interderivability phenomena of seemingly unrelated mathematical statements would remain completely unintelligible for the mathematical empiricist. For surely the sense data (the physical basis or whatever that may be) that would serve as the empirical foundation in the genesis of lattices, topological spaces, vector spaces, cardinality spectra of models of sets of first order sentences, etc. would have to be very different. Moreover, the properties or relations attributed to those entities in the mathematically equivalent statements that we have been considering are very different, and since for a genuine mathematical empiricist they too must be empirically founded, their empirical foundations would also have to diverge. Hence, there is no way for (a genuine) mathematical empiricism to explain the interderivability phenom-

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[13] See Philip Kitcher, *The Nature of Mathematical Knowledge*, Oxford: Oxford University Press, 1983.

[14] Ibid., p. 109.

[15] Ibid., p. 107.

[16] Ibid., pp. 108–109.

ena under discussion. They are for mathematical empiricism as puzzling as for any of the other philosophies of mathematics considered above.

It seems unnecessary to examine every sort of non-Platonist philosophy of mathematics with respect to the interderivability phenomena under discussion. In any other case one can argue essentially in the same way to show the inadequacy of such a philosophy of mathematics to assess the interderivability results of seemingly unrelated mathematical statements. Hence, either such mathematical results are simply incapable of any philosophical assessment, or we have to accept a sort of mathematical Platonism as the correct philosophy of mathematics.

But there can be more than one sort of mathematical Platonism, and even if the second member of the former exclusive disjunction is true, that does not entail that any mathematical Platonism can adequately assess the interderivability phenomena.

§ 3 On Platonisms

For a philosophy of mathematics to be defensible, it has to be complemented by an epistemology of mathematics, i.e. an explanation of how is it that we come to have knowledge about mathematical entities. Constructivist philosophies of mathematics like those of Kant and Brouwer seem almost inseparable from their corresponding epistemologies. The main defect, however, of most Platonist philosophies of mathematics is precisely that they have not developed an accompanying epistemology of mathematics. Thus, even if they were to correctly assess the nature of mathematics and adequately resolve the riddle of mathematical entities, the foundational (not historical) genesis of mathematical knowledge would turn into a new puzzle. Of the defenders of a sort of Platonism in this century, only Husserl seems to have sufficiently developed an epistemology of mathematics.[17]

However, a Platonist philosophy of mathematics not only is in need of an accompanying epistemology of mathematics, but presupposes a semantics appropriate for mathematical statements. Without an adequate

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[17] See E. Husserl, op. cit. in footnote 8, Bd. II. U. VI, Kap. VI, in Husserliana, Vol. XIX/2, Den Haag: Nijhoff, 1984. See also our paper "Husserl's Epistemology of Mathematics and the Foundation of Platonism in Mathematics", in *Husserl Studies*, **4**: 81–102 (1987). A possible exception is Kurt Gödel. See his "What is Cantor's Continuum Problem?", reprinted in P. Benacerraf and H. Putnam, op. cit. in footnote 3, pp. 470–485.

underlying semantics, a Platonist philosophy of mathematics does not seem to go much farther in explaining the interderivability phenomena than the philosophies of mathematics considered in §2 above. Of course, a semantics adequate for mathematical statements and a basically correct epistemology of mathematics are no complete guarantee of the correctness of a philosophy of mathematics, although, taken together, they seem to be necessary conditions both of the correctness and of the rational credibility of a Platonist philosophy of mathematics.

Let us consider briefly Frege's philosophy of mathematics. As is well known, Frege defended both a Platonist and a logicist conception of mathematics (with the exclusion of geometry)—which should be clearly separated from each other. Thus, he not only conceived mathematical entities, e.g. numbers, and logical entities, e.g. thoughts,[18] as having an objective but not spatiotemporally bound existence (Platonism), but he also believed that arithmetical concepts could be defined by means of logical concepts, and arithmetical theorems derived—ultimately— from logical axioms (logicism). Moreover, Frege also developed a theory of reference, according to which statements (i.e. assertive sentences)—whether mathematical or not—, when standing alone or in extensional contexts, refer to a truth value, namely, to the true or to the false. For Frege, the true and the false are not only the referents of all statements, but also in some sense the foremost Platonic entities.[19] Frege does not acknowledge the existence of states of affairs, and for him all true statements have the same reference, namely, the true. Thus, for Frege, the statements (1) 'Paris is the capital of France', (2) '2 + 2 = 4'. (3) 'Every set can be well-ordered', and (4) 'The product of any family of compact topological spaces is a compact topological space', although they seem to speak about very different things, have the same reference, namely, the true. Apart from the fact that statements (2), (3) and (4) seem to be true in all possible worlds, whereas (1) does not, (3) and (4) are mathematically equivalent and are mathematically equivalent neither with (1) nor with (2), nor is (1) mathematically equivalent with (2). Frege's semantics ignores all of this and also runs counter to our intuitions that statements (1)–(4) not only express very different thoughts, but also

[18] See, e.g. "Der Gedanke'" 1918, in Gottlob Frege, *Kleine Schriften*, edited by I. Angelelli, Darmstadt: Wissenschaftliche Buchgesellschaft, 1967, pp. 342–362.

[19] See Frege's *Grundgesetze der Arithmetik*, Bd. I, 1893; Hildesheim: Georg Olms, 1962, § 10.

speak about very different things. Hence, Frege's semantics seems inappropriate for mathematics and, particularly, does not adequately assess the interderivability phenomena of seemingly unrelated mathematical statements, since it does not do justice to the fact that statements (3) and (4) above are interderivable, but are interderivable neither with (1) nor with (2), nor is (1) interderivable with (2).

The first step for building a semantics of mathematics that can adequately assess the interderivability of seemingly unrelated mathematical statements consists in taking states of affairs as the referents of statements.[20] The statements '3 + 4 = 7' and '6 + 1 = 7' express different thoughts, but refer to the same state of affairs, and the statements (1) 'Every filter on a set can be extended to an ultrafilter' and (2) 'Every dual ideal on a set can be extended to a maximal dual ideal' also seem to express different thoughts—if 'filter' is not introduced into the theory as an abbreviation of 'dual ideal'—, but refer to the same state of affairs, namely, to the mathematical fact that an entity with the properties of a filter can be extended to (i.e. is contained in) a filter which is maximal in the sense of not being properly contained in any other filter. Moreover, this state of affairs is clearly different from that referred to by the equations '3 + 4 = 7' and '6 + 1 = 7', although all four statements are true.

But to acknowledge states of affairs as the reference of statements is clearly not enough, since precisely in each of the two lists of mathematically equivalent statements of §1, the statements, although interderivable, speak about very different things and, thus, refer to very different states of affairs. On the other hand, since all statements in both lists have the same truth value as the statements 'Paris is the capital of France' and 'Frege died in 1925', namely, the true, to adequately assess the interderivability of seemingly unrelated mathematical statements, a semantics appropriate for mathematics has to postulate the existence of abstract entities intermediate between states of affairs and truth values. Thus, we will speak of abstract situations of affairs, and will say that the statements in each of the two lists of mathematically equivalent statements of §1 have the same abstract situation of affairs as their reference basis, although the states of affairs referred to by them are clearly different. More simple examples of pairs of mathematical statements referring to

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20 Compare the rest of this § with our paper "On Frege's Two Notions of Sense", in *History and Philosophy of Logic*, 7: 31–41 (1986). See also our "Remarks on Sense and Reference in Frege and Husserl", in *Kantstudien*, **73**: 425–439 (1982).

different states of affairs but having the same abstract situation of affairs as reference basis are the somewhat trivial pair of inequalities. '5 > 3' and '3 < 5', and pairs of dual statements as, e.g. the pair (1) 'Every filter on a set can be extended to an ultrafilter' and (2) 'Every ideal on a set can be extended to a maximal ideal'. (That "abstract situations of affairs" are really "abstract" can be easily admitted if we try to apprehend them intuitively even in the latter more elementary examples considered above.)

On the other hand, the statements 'Paris is the capital of France', 'Frege died in 1925' and '2 + 2 = 4' have very different abstract situations of affairs as their reference bases, and all these reference bases differ both from the common abstract situation of affairs of, e.g. the Axiom of Choice and Tychonoff's Theorem, and from the common abstract situation of affairs of, e.g. the Ultrafilter Theorem and the Compactness Theorem. Thus, a semantics for mathematical statements that can offer an appropriate assessment of mathematical equivalence must include the following schema (where the arrows represent functions):

Statement

↓

Thought (or Proposition)

↓

State of Affairs (= Reference of Statements)

↓

Abstract Situation of Affairs (= Reference Basis of Statements)

↓

Truth Value

This semantics for statements is essentially a reconstruction of Husserl's, as is the distinction—and the terminology—between state of affairs (*Sachverhalt*) and situation of affairs (*Sachlage*).[21] This distinction remains in Husserl's writings somewhat sketchy, and since we are using

[21] E, Husserl, op. cit. in footnote 8, Bd. II, U. VI, § 48 and especially *Erfahrung und Urteil*, 1939, fifth revised edition, Hamburg: Meiner, 1976, §§ 58–65.

it exclusively for mathematical contexts, for which the distinction is much clearer and can be made much more precise than for non-mathematical ones, we have added the adjective 'abstract' to 'situation of affairs' not so much to emphasize its abstract character—which is present even in non-mathematical contexts—[22], as to underscore that the notion of abstract situation of affairs that we have introduced here for a purpose not explicitly envisaged by Husserl, should be regarded more as a sort of explanans of Husserl's notion of situation of affairs than as exactly the same notion. On the other hand, since Husserl also developed an epistemology of mathematics, combining it with the neo-Husserlian semantics of mathematics just sketched, it seems possible to defend mathematical Platonism along neo-Husserlian lines.

Finally, it seems interesting to examine the semantics of mathematical statements propounded in this paper from the point of view of the information conveyed. As is well known, Frege begins and ends his famous 'Über Sinn und Bedeutung' with a discussion of identity statements.[23] In particular, he is interested in explaining how is it that an identity statement of the form '$a = b$', when true—no matter whether synthetically or analytically—, can have a greater cognitive value and, thus, be more informative, than an identity statement of the form '$a = a$'. The identity statements '117 = 117' and '117 = 9 × 13' are both true identity statements and, according to Frege, analytically true, but the second is much more informative than the first. To explain this situation, Frege introduces the notion of sense. '117' and '9 × 13' have the same reference, namely, the number 117, but their senses are different and, thus, the senses of the identity statements '117 = 117' and '117 = 9 × 13' are different.[24] Thus, if in a mathematical (or non-mathematical) statement, standing alone or in an extensional context, we substitute a proper name (in its wide Fregean meaning) for another proper name with a different sense but with the same reference, the truth value of the statement remains the same but its cognitive value can change.

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[22] See e.g. *Logische Untersuchungen*, Bd. II, U. VI, § 48. See also our paper cited in footnote 20.

[23] 'Über Sinn und Bedeutung', 1982, in *Kleine Schriften*, pp. 143–162. See also our 'Identity Statements in the Semantics of Sense and Reference', in *Logique et Analyse*, **25**: 399–411 (1982).

[24] It is not clear what the sense of the expression '117' actually is. We can assume, for simplicity's sake, that it is the same as that of the expression '116 + 1'. In any case, it seems to be clearly different from that of the expression '9 × 13'.

However, a mathematical statement conveys information not only at the level of senses, but also at the level of states of affairs. Mathematical (and non-mathematical) statements refer to states of affairs, and when someone learns to what state of affairs does a mathematical statement refer, he obtains some information. When a mathematical statement asserts that a mathematical entity, e.g. an ultraproduct or a Hausdorff space, has a definite property, it conveys (a non-trivial) information, based on an understanding of the referents of the constituents of the statement, namely, ultraproducts, Hausdorff spaces, etc.

Moreover, at the level of abstract situations of affairs there seems to lie a deeper, more intangible and probably less universal[25] level of information which has a strong metamathematical flavor.[26] When someone grasps a statement that speaks about the interderivability of two seemingly unrelated mathematical statements, a certain information is conveyed to him, an information at a level which builds a sort of 'deep structure' of mathematics.

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[25] Interderivability phenomena like those considered in this paper seem to be rather isolated phenomena. That for any mathematical statement, there are other mathematical statements that refer to different states of affairs but are mathematically equivalent to it, seems improbable and, in any case, would have to be proved.

[26] It should be clear from the very beginning of this paper that the interderivability results have a metamathematical character. We have not emphasized this point to avoid somewhat esoteric terminology that could originate unnecessary confusion.